Listy Biometryczne - Biometrical Letters
Vol. 35(1998), No. 1, 11-26

On admissibility of the intra-block and inter-block variance component estimators

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SUMMARY

In the paper the problem of admissibility of two kinds of estimators of the variance components appearing in the randomization model used for experiments in block designs is considered. It is proved, that the intra-block estimator of the variance of errors is admissible in the class of unbiased estimators. Sufficient conditions are given, for which the inter-block estimator of the variance of block effects is admissible. The conditions are given in terms of the incidence matrix of the design. Some examples of models for which this inter-block estimator is inadmissible are presented. In such cases it is shown how to improve uniformly the estimator by using one of the admissible estimators.

KEY WORDS: admissibility, block designs, estimators of variance components, inter- and intra-block estimation, invariant quadratic unbiased estimators.

1. Introduction

For the two-way classification model corresponding to a block design with random block effects, a problem of interest is to estimate two variance components, the variance of block effects and the variance of errors. As it has been established by Baksalary et al. (1990), the existence of the uniformly minimum variance unbiased estimators is assured only for special block designs. Generally there is a rich choice of invariant quadratic and unbiased estimators, each of them being admissible with respect to the quadratic loss. Since there is no preference for any of the admissible

*Research partially supported by Komitet Badań Naukowych, PB 2 PO3A 067 10.

estimators to be used in practical situations, some additional criteria are taken into account. Caliński and Kageyama (1991) have considered unbiased variance component estimators based on the intra-block and on the inter-block analysis. The proposed estimators are uniquelly given and have, under the corresponding submodels, the desirable MINQUE properties (Sections 3.1 and 3.2 of their paper). However, in some special cases (as those considered in Section 3.4 of that paper), estimation under the overall randomization model is relevant. The question then arises whether the variance component estimators derived from the submodels are admissible in the class of all invariant quadratic and unbiased estimators under the overall model. This is the main problem considered in the present paper.

2. Mixed model corresponding to a block design

Consider an experiment in which v treatments are applied to n experimental units arranged in b blocks according to a $v \times b$ incidence matrix $\mathbf{N}$ with entries $n_{ij} \geq 0$. Here n_{ij} is the number of units corresponding to the (i, j)-th cell of the matrix, i.e., treated by the i-th treatment and belonging to the j-th block. Let y_{ijl} be the observation taken on the l-th unit of the (i, j)-th cell, $l = 1, 2, ..., n_{ij}$.

One of the basic assumptions is the additivity of the effects of the treatments and those of the units, and also of possible technical errors, which can be written for $i = 1, 2, ..., v$, $j = 1, 2, ..., b$ and $l = 1, 2, ..., n_{ij}$, as

$$y_{ijl} = \tau_i + \beta_j + \varepsilon_{ijl}. \tag{1}$$

Here τ_i is the effect of the i-th treatment, β_j is the effect of the j-th block, while ε_{ijl}'s are disturbances including unit and technical errors. As it has been established in a series of papers by Kala (1989, 1990, 1991) and also in the papers by Caliński and Kageyama (1991, 1996), following the randomization theory, the expectation and the covariance matrix of the observed $n \times 1$ vector $\mathbf{y}$ can be presented as

$$E(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau},$$

$$\begin{aligned}\mathrm{Cov}(\mathbf{y}) &= (\mathbf{D}'\mathbf{D} - N_B^{-1}\mathbf{1}_n\mathbf{1}_n')\sigma_B^2 + (\mathbf{I}_n - K_H^{-1}\mathbf{D}'\mathbf{D})\sigma_u^2 + \mathbf{I}_n\sigma_e^2 \\ &= \mathbf{D}'\mathbf{D}\sigma_1^2 + \mathbf{I}_n\sigma^2 - N_B^{-1}\mathbf{1}_n\mathbf{1}_n'\sigma_B^2,\end{aligned} \tag{2}$$

where $\sigma_1^2 = \sigma_B^2 - K_H^{-1}\sigma_u^2$ and $\sigma^2 = \sigma_u^2 + \sigma_e^2$. Here $\boldsymbol{\Delta}'$ and $\mathbf{D}'$ are the $n \times v$ and $n \times b$ known design matrices of full ranks v and b, respectively, the elements of which are 0 or 1 depending on the ordering of the components of $\mathbf{y}$. Anyway, it can be seen that $\boldsymbol{\Delta}\mathbf{D}' = \mathbf{N}, \boldsymbol{\Delta}'\mathbf{1}_v = \mathbf{D}'\mathbf{1}_b = \mathbf{1}_n, \boldsymbol{\Delta}\mathbf{1}_n = \mathbf{N}\mathbf{1}_b = \mathbf{r} = (r_1, r_2, ..., r_v)', \mathbf{D}\mathbf{1}_n = \mathbf{N}'\mathbf{1}_v = \mathbf{k} = (k_1, k_2, ..., k_b)'$, while $\boldsymbol{\Delta}\boldsymbol{\Delta}' = \mathrm{diag}\{r_i\}$ and $\mathbf{D}\mathbf{D}' = \mathrm{diag}\{k_j\}$ are diagonal matrices with the diagonal elements $r_i = \sum_j n_{ij}$ and $k_j = \sum_i n_{ij}$, respectively (cf. Caliński, 1993, p. 284). The variance components σ_B^2, σ_u^2 and σ_e^2 are assumed to be unknown, σ_B^2 represents the variance of block effects, while $\sigma^2 = \sigma_u^2 + \sigma_e^2$ is the common variance of the disturbances ε_{ijl}, σ_u^2 being the variance of unit errors and σ_e^2 that of technical errors. The above model, subsequently called the overall model with the covariance matrix (2), follows from the basic principles of randomization of the units within N_B available blocks, and of randomized selection of b of them for the experiment. Here K_H is a weighted harmonic average of the numbers of units within blocks (defined in Caliński and Kageyama, 1991, p.100).

3. Estimation of variance components

3.1. Submodels

Following Caliński and Kageyama (1991), let the observed vector $\mathbf{y}$ be decomposed as $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3$, where $\mathbf{y}_i$ is the orthogonal projection of $\mathbf{y}$ on $\mathbb{R}_i$, $i = 1, 2, 3$, defined as

$$\mathbb{R}^n = \mathbb{R}_1 + \mathbb{R}_2 + \mathbb{R}_3,$$

where

$$\mathbb{R}_1 = \mathcal{N}(\mathbf{D}),\ \mathbb{R}_2 = \mathcal{N}(\mathbf{1}_n') \cap \mathcal{R}(\mathbf{D}'),\ \mathbb{R}_3 = \mathcal{R}(\mathbf{1}_n).$$

Here for a given matrix $\mathbf{A}$ the symbol $\mathcal{R}(\mathbf{A})$ stands for the range space of $\mathbf{A}$, while $\mathcal{N}(\mathbf{A})$ for the kernel (null space) of $\mathbf{A}$.

Since the orthogonal projectors on $\mathbb{R}_i$ are, respectively, $\mathbf{\Phi}_1 = \mathbf{I}_n - \mathbf{D}'(\mathbf{D}\mathbf{D}')^{-1}\mathbf{D}$, $\mathbf{\Phi}_2 = \mathbf{D}'(\mathbf{D}\mathbf{D}')^{-1}\mathbf{D} - n^{-1}\mathbf{1}_n\mathbf{1}_n'$, and $\mathbf{\Phi}_3 = n^{-1}\mathbf{1}_n\mathbf{1}_n'$, the decomposition leads to the following three independent submodels:

(i) intra-block model: $E(\mathbf{y}_1) = \mathbf{\Phi}_1\mathbf{\Delta}'\boldsymbol{\tau}$, with

$$\mathrm{Cov}(\mathbf{y}_1) = \sigma^2\mathbf{\Phi}_1,\ \sigma^2 = \sigma_u^2 + \sigma_e^2,$$

(ii) inter-block model: $E(\mathbf{y}_2) = \mathbf{\Phi}_2\mathbf{\Delta}'\boldsymbol{\tau}$, with

$$\mathrm{Cov}(\mathbf{y}_2) = \sigma_1^2\mathbf{\Phi}_2\mathbf{D}'\mathbf{D}\mathbf{\Phi}_2 + \sigma^2\mathbf{\Phi}_2,\ \sigma_1^2 = \sigma_B^2 - K_H^{-1}\sigma_u^2,$$

(iii) total area model: $E(\mathbf{y}_3) = \mathbf{\Phi}_3\mathbf{\Delta}'\boldsymbol{\tau}$, with

$$\mathrm{Cov}(\mathbf{y}_3) = [(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 - n^{-1}\mathbf{k}'\mathbf{k}K_H^{-1}\sigma_u^2 + \sigma^2]\mathbf{\Phi}_3.$$

3.2. Projectors

For the intra- and inter-block model the squared norm of the vector $\mathbf{y}_i$, $i = 1, 2$, can be decomposed as

$$\mathbf{y}_i'\mathbf{y}_i = \mathbf{y}'\mathbf{\Phi}_i\mathbf{y} = \mathbf{y}'\mathbf{\Phi}_i\mathbf{\Delta}'\mathbf{C}_i^+\mathbf{\Delta}\mathbf{\Phi}_i\mathbf{y} + \mathbf{y}'\mathbf{\Phi}_i(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_i^+\mathbf{\Delta})\mathbf{\Phi}_i\mathbf{y},$$

where $\mathbf{C}_i = \mathbf{\Delta}\mathbf{\Phi}_i\mathbf{\Delta}'$. Here $\mathbf{C}_i^+$ stands for the Moore-Penrose inverse of $\mathbf{C}_i$, and can be replaced by any generalized inverse (g-inverse) of it, $\mathbf{C}_i^-$, $i = 1, 2$. Denote

$$\mathbf{\Pi}_i = \mathbf{\Phi}_i(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_i^+\mathbf{\Delta})\mathbf{\Phi}_i, \ i = 1, 2.$$

LEMMA 3.1.
(i) $\mathbf{\Pi}_1$ *is the orthogonal projector on* $\mathcal{N}(\mathbf{D}) \cap \mathcal{N}(\mathbf{\Delta})$,
(ii) $\mathbf{\Pi}_2$ *is the orthogonal projector on* $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta})$.

The proof of Lemma 3.1 follows from the following proposition.

PROPOSITION 3.1. *If* $\mathbf{\Phi}$ *is the orthogonal projector on an arbitrary subspace* $\mathcal{E}$ *of* $\mathbb{R}^n$, *then, for a given matrix* $\mathbf{A}$ *such that* $\mathcal{N}(\mathbf{A}) \subseteq \mathbb{R}^n$, *and* $\mathbf{C} = \mathbf{A}\mathbf{\Phi}\mathbf{A}'$,

$$\mathbf{\Pi} = \mathbf{\Phi}(\mathbf{I}_n - \mathbf{A}'\mathbf{C}^+\mathbf{A})\mathbf{\Phi}$$

is the orthogonal projector on $\mathcal{E} \cap \mathcal{N}(\mathbf{A})$.

Proof of Proposition 3.1. Let $\mathbf{a} \in \mathcal{E} \cap \mathcal{N}(\mathbf{A})$, then $\mathbf{\Phi a} = \mathbf{a}$, $\mathbf{Aa} = \mathbf{0}$ and in consequence $\mathbf{\Pi a} = \mathbf{a}$. For $\mathbf{b} \in [\mathcal{E} \cap \mathcal{N}(\mathbf{A})]^\perp = \mathcal{E}^\perp + \mathcal{R}(\mathbf{A}')$ we have $\mathbf{b} = \mathbf{c} + \mathbf{A}'\mathbf{d}$, $\mathbf{c} \in \mathcal{E}^\perp$ ($\mathbf{\Phi c} = \mathbf{0}$), and $\mathbf{\Pi b} = \mathbf{\Pi A}'\mathbf{d} = \mathbf{\Phi A}'\mathbf{d} - \mathbf{\Phi A}'(\mathbf{A\Phi A}')^+\mathbf{A\Phi A}'\mathbf{d} = \mathbf{0}$. The last equality follows from the fact that $\mathbf{\Phi A}'(\mathbf{A\Phi A}')^+\mathbf{A\Phi A}' = \mathbf{\Phi A}'$ (cf. Rao and Mitra 1971, Lemma 2.2.6). □

Proof of Lemma 3.1. We find from Proposition 3.1 that $\mathbf{\Pi}_1$ is the orthogonal projector on $\mathcal{N}(\mathbf{D}) \cap \mathcal{N}(\mathbf{\Delta})$, while $\mathbf{\Pi}_2$ is the orthogonal projector on $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{1}_n') \cap \mathcal{N}(\mathbf{\Delta})$. Since $\mathbf{1}_n \in \mathcal{R}(\mathbf{\Delta}')$, it follows that $\mathcal{N}(\mathbf{\Delta}) \subset \mathcal{N}(\mathbf{1}_n')$ and $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{1}_n') \cap \mathcal{N}(\mathbf{\Delta}) = \mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta})$. □

It follows from Propositon 3.1 and Lemma 3.1 that equivalent forms of $\mathbf{\Pi}_i$, $i = 1, 2$, are

$$\mathbf{\Pi}_1 = \mathbf{M}_\Delta(\mathbf{I}_n - \mathbf{D}'\mathbf{C}_\Delta^+\mathbf{D})\mathbf{M}_\Delta, \tag{3}$$

where $\mathbf{M}_\Delta = \mathbf{I}_n - \mathbf{\Delta}'(\mathbf{\Delta\Delta}')^{-1}\mathbf{\Delta}$ is the orthogonal projector on $\mathcal{N}(\mathbf{\Delta})$, while $\mathbf{C}_\Delta = \mathbf{D}\mathbf{M}_\Delta\mathbf{D}'$, and

$$\mathbf{\Pi}_2 = \mathbf{\Phi}_{23}(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_{23}^+\mathbf{\Delta})\mathbf{\Phi}_{23}, \tag{4}$$

where $\mathbf{\Phi}_{23} = \mathbf{D}'(\mathbf{DD}')^{-1}\mathbf{D} = \mathbf{\Phi}_2 + \mathbf{\Phi}_3$ is the orthogonal projector on $\mathcal{R}(\mathbf{D}')$, while $\mathbf{C}_{23} = \mathbf{\Delta\Phi}_{23}\mathbf{\Delta}' = \mathbf{C}_2 + n^{-1}\mathbf{\Delta}\mathbf{1}_n\mathbf{1}_n'\mathbf{\Delta}' = \mathbf{C}_2 + n^{-1}\mathbf{rr}'$.

3.3. Intra- and inter-block estimators

Let us consider the quadratic forms

$$\mathbf{y}'\mathbf{\Pi}_1\mathbf{y} = \mathbf{y}'\mathbf{\Phi}_1(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_1^+\mathbf{\Delta})\mathbf{\Phi}_1\mathbf{y} = \mathbf{y}'\mathbf{M}_\Delta(\mathbf{I}_n - \mathbf{D}'\mathbf{C}_\Delta^+\mathbf{D})\mathbf{M}_\Delta\mathbf{y}$$

and

$$\mathbf{y}'\mathbf{\Pi}_2\mathbf{y} = \mathbf{y}'\mathbf{\Phi}_2(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_2^+\mathbf{\Delta})\mathbf{\Phi}_2\mathbf{y} = \mathbf{y}'\mathbf{\Phi}_{23}(\mathbf{I}_n - \mathbf{\Delta}'\mathbf{C}_{23}^+\mathbf{\Delta})\mathbf{\Phi}_{23}\mathbf{y}.$$

It can easily be found that $\mathbf{\Pi}_i\mathbf{\Delta}' = \mathbf{0}$ and $\mathbf{\Pi}_i\mathbf{1}_n = \mathbf{0}$ for $i = 1, 2$, and $\mathbf{\Pi}_1\mathbf{D}' = \mathbf{0}$. It follows that $\mathbf{y}'\mathbf{\Pi}_i\mathbf{y}$ are invariant with respect to the mean vector translations, i.e. $\mathbf{y}'\mathbf{\Pi}_i\mathbf{y} = (\mathbf{y} - \mathbf{\Delta}'\boldsymbol{\alpha})'\mathbf{\Pi}_i(\mathbf{y} - \mathbf{\Delta}'\boldsymbol{\alpha})$ for any $\boldsymbol{\alpha} \in \mathbb{R}^{\mathrm{v}}$, and

$$E(\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}) = \mathrm{rank}(\mathbf{\Pi}_1)\sigma^2,$$

where $\mathrm{rank}(\mathbf{\Pi}_1) = \dim\{\mathcal{N}(\mathbf{D}) \cap \mathcal{N}(\mathbf{\Delta})\} = n - b - \mathrm{rank}(\mathbf{C}_1) = n - v - \mathrm{rank}(\mathbf{C}_\Delta)$,

$$E(\mathbf{y}'\mathbf{\Pi}_2\mathbf{y}) = \mathrm{tr}(\mathbf{D}\mathbf{\Pi}_2\mathbf{D}')\sigma_1^2 + \mathrm{rank}(\mathbf{\Pi}_2)\sigma^2,$$

where $\mathrm{rank}(\mathbf{\Pi}_2) = \dim\{\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta})\} = b - 1 - \mathrm{rank}(\mathbf{C}_2) = b - \mathrm{rank}(\mathbf{N})$ (cf. Caliński and Kageyama, 1991, Sections 3.1 and 3.2).

Thus $\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}$ and $\mathbf{y}'\mathbf{\Pi}_2\mathbf{y}$ are unbiased estimators for $\mathrm{rank}(\mathbf{\Pi}_1)\sigma^2$ and $\mathrm{tr}(\mathbf{D}\mathbf{\Pi}_2\mathbf{D}')\sigma_1^2 + \mathrm{rank}(\mathbf{\Pi}_2)\sigma^2$, respectively. The estimators are based on the intra- and inter-block analysis, respectively. In consequence,

$$\hat{\sigma}^2 = \frac{\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}}{\mathrm{rank}(\mathbf{\Pi}_1)} \tag{5}$$

and

$$\hat{\sigma}_1^2 = \frac{\mathbf{y}'\mathbf{\Pi}_2\mathbf{y} - \mathrm{rank}(\mathbf{\Pi}_2)\hat{\sigma}^2}{\mathrm{tr}(\mathbf{D}\mathbf{\Pi}_2\mathbf{D}')} \tag{6}$$

are invariant quadratic unbiased estimators of σ^2 and σ_1^2, respectively, with $\mathrm{tr}(\mathbf{D}\mathbf{\Pi}_2\mathbf{D}') = \mathrm{rank}(\mathbf{\Pi}_2)k$ if $k_1 = k_2 = ... = k_b = k$ (say). The problem that now arises is whether these estimators are admissible within the class of all invariant quadratic and unbiased estimators in the overall model with the covariance matrix (2). This problem is particularly interesting from the point of view of estimating variances of some linear estimators considered in Corollary 3.3(a) and (b) of Caliński and Kageyama (1991), viz. best linear unbiased estimators (BLUEs) of contrasts that can be estimated exclusively either in the intra-block or in the inter-block analysis.

4. Admissibility of $\hat{\sigma}^2$ and $\hat{\sigma}_1^2$

4.1. The set-up

At the begining we recall the main results concerning admissible invariant quadratic and unbiased estimators that are applicable to the overall model considered in Section 2. The results come from Olsen et al. (1976), Gnot and Kleffe (1983) and Gnot at al. (1992). We will be considering estimators that are invariant under the group of transformations $\mathbf{y} \to \mathbf{y} + \boldsymbol{\Delta}'\mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^v$. Thus we restrict the considerations to quadratic forms $\mathbf{y}'\mathbf{A}\mathbf{y}$, for which $\mathbf{A}\boldsymbol{\Delta}' = \mathbf{0}$. Since $\mathbf{1}_n \in \mathbb{R}(\boldsymbol{\Delta}')$, we have $\mathbf{A}\mathbf{1}_n = \mathbf{0}$ for an invariant estimator $\mathbf{y}'\mathbf{A}\mathbf{y}$. It follows that instead of the model with the covariance matrix (2) we can consider a simpler model, with

$$E(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau},$$

$$\mathrm{Cov}(\mathbf{y}) = \sigma_1^2\mathbf{D}'\mathbf{D} + \sigma^2\mathbf{I}_n. \tag{7}$$

Results obtained under (7) will then apply to the original model with (2) as well (see also Caliński and Kageyama, 1996, Lemma 3.1). Following Olsen at al. (1976) a maximal invariant statistic with respect to the mean vector translations is $\mathbf{t} = \mathbf{B}\mathbf{y}$, where $\mathbf{B}$ is an $(n-v) \times n$ matrix such that $\mathbf{B}\mathbf{B}' = \mathbf{I}_{n-v}$ and $\mathbf{B}'\mathbf{B} = \mathbf{M}_{\boldsymbol{\Delta}}$.

Denote by $\alpha_1 > ... > \alpha_{d-1} > \alpha_d = 0$ the ordered sequence of the different eigenvalues of $\mathbf{W} = \mathbf{B}\mathbf{D}'\mathbf{D}\mathbf{B}'$, of rank equal to that of $\mathbf{C}_{\boldsymbol{\Delta}} = \mathbf{D}\mathbf{B}'\mathbf{B}\mathbf{D}'$. Let $\mathbf{W} = \sum_{i=1}^{d-1} \alpha_i\mathbf{E}_i$ be the spectral decomposition of the matrix $\mathbf{W}$. Next, consider a random vector $\mathbf{z} = (z_1, z_2, ..., z_d)'$, with $z_i = \mathbf{t}'\mathbf{E}_i\mathbf{t}/\nu_i$. Here $\nu_1, \nu_2, ..., \nu_d$ are the multiplicities of α_i's, where $\alpha_d = 0$ with $\nu_d = n - v - \mathrm{rank}(\mathbf{C}_{\boldsymbol{\Delta}}) = \mathrm{rank}(\boldsymbol{\Pi}_1) > 0$. Under normality of $\mathbf{y}$ the random variables z_i are independent, and $\nu_i z_i/(\alpha_i\sigma_1^2 + \sigma^2)$ has a central chi-square distribution with ν_i degrees of freedom, $i = 1, 2, ..., d$. Consider again the matrix

$$\mathbf{C}_{\boldsymbol{\Delta}} = \mathbf{D}\mathbf{M}_{\boldsymbol{\Delta}}\mathbf{D}' = \mathbf{D}\mathbf{D}' - \mathbf{N}'(\boldsymbol{\Delta}\boldsymbol{\Delta}')^{-1}\mathbf{N}.$$

Since $\mathbf{C}_{\boldsymbol{\Delta}} = \mathbf{D}\mathbf{B}'\mathbf{B}\mathbf{D}'$, it follows that the positive eigenvalues of $\mathbf{C}_{\boldsymbol{\Delta}}$ and $\mathbf{W}$ are the same. Let

$$\mathbf{C}_{\boldsymbol{\Delta}} = \sum_{i=1}^{d-1} \alpha_i\mathbf{C}_{\boldsymbol{\Delta}i}$$

be the spectral decomposition of $\mathbf{C}_{\boldsymbol{\Delta}}$. Using the fact that if $\mathbf{w}$ is a normalized eigenvector of $\mathbf{W}$ corresponding to a positive eigenvalue α, then $\frac{1}{\sqrt{\alpha}}\mathbf{D}\mathbf{B}'\mathbf{w}$ is a normalized eigenvector of $\mathbf{C}_{\boldsymbol{\Delta}}$ corresponding to that α, we find that

$$\nu_i z_i = \mathbf{y}'\mathbf{B}'\mathbf{E}_i\mathbf{B}\mathbf{y} = \frac{1}{\alpha_i}\mathbf{y}'\mathbf{M}_{\boldsymbol{\Delta}}\mathbf{D}'\mathbf{C}_{\boldsymbol{\Delta}i}\mathbf{D}\mathbf{M}_{\boldsymbol{\Delta}}\mathbf{y}, \quad i = 1, 2, ..., d-1,$$

$$\sum_{i=1}^{d-1} \nu_i z_i = \mathbf{y}'\mathbf{M}_\Delta \mathbf{D}'\mathbf{C}_\Delta^+ \mathbf{D}\mathbf{M}_\Delta \mathbf{y}, \quad \sum_{i=1}^{d} \nu_i z_i = \mathbf{y}'\mathbf{M}_\Delta \mathbf{y}, \tag{8}$$

and hence,

$$\nu_d z_d = \mathbf{y}'\mathbf{M}_\Delta(\mathbf{I}_n - \mathbf{D}'\mathbf{C}_\Delta^+\mathbf{D})\mathbf{M}_\Delta \mathbf{y} = \mathbf{y}'\boldsymbol{\Pi}_1\mathbf{y}, \quad \nu_d = \operatorname{rank}(\boldsymbol{\Pi}_1) \tag{9}$$

(cf. Gnot et al., 1992, Section 2). Let the space $\mathcal{N}(\boldsymbol{\Delta})$ be decomposed as

$$\mathcal{N}(\boldsymbol{\Delta}) = \mathcal{N}(\mathbf{D}) \cap \mathcal{N}(\boldsymbol{\Delta}) + \mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta}),$$

and similarly the orthogonal projector $\mathbf{M}_\Delta$ on $\mathcal{N}(\boldsymbol{\Delta})$ as

$$\mathbf{M}_\Delta = \boldsymbol{\Pi}_1 + \mathbf{M}_\Delta \mathbf{D}'\mathbf{C}_\Delta^+ \mathbf{D}\mathbf{M}_\Delta,$$

with $\mathbf{M}_\Delta \mathbf{D}'\mathbf{C}_\Delta^+ \mathbf{D}\mathbf{M}_\Delta$ being the orthogonal projector on $\mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta})$.

Remark 4.1. Note that

$$\dim\{\mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta})\} = \operatorname{rank}(\mathbf{M}_\Delta \mathbf{D}'\mathbf{C}_\Delta^+ \mathbf{D}\mathbf{M}_\Delta) = \operatorname{rank}(\mathbf{C}_\Delta),$$

while $\boldsymbol{\Pi}_2$ projects on

$$\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\boldsymbol{\Delta}) \subseteq \mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta}),$$

and

$$\dim\{\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\boldsymbol{\Delta})\} = \operatorname{rank}(\boldsymbol{\Pi}_2) =$$
$$= \operatorname{tr}(\boldsymbol{\Phi}_{23}) - \operatorname{tr}(\boldsymbol{\Delta}'\mathbf{C}_{23}^+\boldsymbol{\Delta}\boldsymbol{\Phi}_{23}) = b - \operatorname{rank}(\mathbf{N}). \tag{10}$$

It follows then that

(i) $\boldsymbol{\Pi}_2 \neq 0$ iff $\operatorname{rank}(\mathbf{N}) < b$,
(ii) $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\boldsymbol{\Delta}) = \mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta})$ iff $\operatorname{rank}(\mathbf{N}) = b - \operatorname{rank}(\mathbf{C}_\Delta)$,

and in such a case $\boldsymbol{\Pi}_2$ becomes equal to $\mathbf{M}_\Delta \mathbf{D}'\mathbf{C}_\Delta^+ \mathbf{D}\mathbf{M}_\Delta$, i.e.,

$$\boldsymbol{\Pi}_2 \text{ projects on } \mathcal{R}(\mathbf{D}' : \boldsymbol{\Delta}') \cap \mathcal{N}(\boldsymbol{\Delta}),$$

$$\operatorname{rank}(\boldsymbol{\Pi}_2) = \operatorname{rank}(\mathbf{C}_\Delta),$$

and

$$\operatorname{tr}(\mathbf{D}\boldsymbol{\Pi}_2\mathbf{D}') = \operatorname{tr}(\mathbf{C}_\Delta).$$

Under the above assumption, as it has been mentioned by Olsen et al. (1976 p. 889), $\hat{\sigma}_1^2$ coincides with the Henderson III estimator (i.e., obtained by Henderson's Method III) and in this case some results of Proposition 6.2 in Olsen et al. (1976) can be applied to establish admissibility of $\hat{\sigma}_1^2$.

We assume throughout the paper that $\mathbf{\Pi}_2 \neq \mathbf{0}$.

It follows from Gnot and Kleffe (1983) that for a given function $f_1\sigma_1^2 + f_2\sigma^2$ the entire class of admissible invariant quadratic unbiased estimators in the present model, that with (7), coincides with the linear combinations of z_i of the form

$$\bar{\gamma}(u,v) = \sum_{i=1}^{d-1}(\lambda_1\alpha_i + \lambda_2)\nu_i w_i(u,v) z_i + \lambda_2\nu_d z_d, \quad u,v \geq 0, \tag{11}$$

where $w_i(u,v) = [1 + 2u\alpha_i + (u^2+v)\alpha_i^2]^{-1}$, or

$$\bar{\gamma}(\infty) = \sum_{i=1}^{d-1} \frac{\lambda_1\nu_i}{\alpha_i} z_i + \lambda_2\nu_d z_d. \tag{12}$$

Here λ_1 and λ_2 are chosen such that $\bar{\gamma}(u,v)$ or $\bar{\gamma}(\infty)$ is unbiased for $f_1\sigma_1^2 + f_2\sigma^2$, i.e.,

$$\sum_{i=1}^{d-1}(\lambda_1\alpha_i + \lambda_2) w_i(u,v)\alpha_i\nu_i = f_1 \tag{13}$$

and

$$\sum_{i=1}^{d-1}(\lambda_1\alpha_i + \lambda_2) w_i(u,v)\nu_i + \lambda_2\nu_d = f_2 \tag{14}$$

for $\bar{\gamma}(u,v)$, while

$$\lambda_1 \text{rank}(\mathbf{C}_\Delta) = f_1, \quad \lambda_1 \sum_{i=1}^{d-1} \frac{\nu_i}{\alpha_i} + \lambda_2\nu_d = f_2 \tag{15}$$

for $\bar{\gamma}(\infty)$. The estimator $\bar{\gamma}(u,v)$ is a Bayesian invariant quadratic unbiased estimator (IQUE) with respect to the prior distribution ρ on (σ_1^2, σ^2) such that $E_\rho(\sigma_1^2) = u$, $E_\rho(\sigma^2) = 1$, $\text{Var}_\rho(\sigma_1^2) = v$, $\text{Var}_\rho(\sigma^2) = 0$, while $\bar{\gamma}(\infty)$ is obtained from (11) as a unique limit of $\bar{\gamma}(u,v)$ if v tends to infinity.

4.2. Admissibility of $\hat{\sigma}^2$

LEMMA 4.1. *The intra-block estimator* (5), *i.e.*

$$\hat{\sigma}^2 = \frac{\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}}{\text{rank}(\mathbf{\Pi}_1)},$$

is admissible in the overall model, with (2), *among all invariant quadratic and unbiased estimators of* σ^2.

Proof. Taking $f_1 = 0$ and $f_2 = 1$ in (15), it follows from (9) and (12) that $\hat{\sigma}^2 = z_d = \hat{\gamma}(\infty)$ for σ^2, which establishes Lemma 4.1. □

The admissibility of $\hat{\sigma}^2 = z_d$ has been proved in a slightly different way by Olsen et al. [1976, Proposition 6.2(a)]. In fact $\hat{\sigma}^2$ is the Henderson III estimator of σ^2.

4.3. Admissibility of $\hat{\sigma}_1^2$

The problem of admissibility of the inter-block estimator (6), i.e., of

$$\hat{\sigma}_1^2 = \frac{y'\mathbf{\Pi}_2 y - \operatorname{rank}(\mathbf{\Pi}_2)\hat{\sigma}^2}{\operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')},$$

is more complicated. Since $\mathbf{\Pi}_2$ projects on $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta}) \subseteq \mathcal{R}(\mathbf{D}' : \mathbf{\Delta}') \cap \mathcal{N}(\mathbf{\Delta})$, and $\mathbf{M_\Delta D'C_\Delta^+ DM_\Delta}$ projects on $\mathcal{R}(\mathbf{D}' : \mathbf{\Delta}') \cap \mathcal{N}(\mathbf{\Delta})$ (see Remark 4.1), it follows from (8), (11) and (12) that if $\hat{\sigma}_1^2$ is admissible, then

$$\hat{\sigma}_1^2 = \frac{1}{\operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')}\sum_{i=1}^{d-1} \nu_i c_i z_i - \frac{\operatorname{rank}(\mathbf{\Pi}_2)}{\operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')} z_d,$$

where $c_i = 0$ or 1, and $c_i = 1$ for each i iff $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta}) = \mathcal{R}(\mathbf{D}' : \mathbf{\Delta}') \cap \mathcal{N}(\mathbf{\Delta})$.

From (11) and (12) we find that

$$\lambda_2 = -\frac{\operatorname{rank}(\mathbf{\Pi}_2)}{\nu_d \operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')} \tag{16}$$

and

$$(\lambda_1\alpha_i + \lambda_2)w_i(u,v) = \frac{c_i}{\operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')} \tag{17}$$

for some $u, v \geq 0$, $\quad i = 1, ..., d-1$, or

$$\frac{\lambda_1}{\alpha_i} = \frac{c_i}{\operatorname{tr}(\mathbf{D\Pi}_2\mathbf{D}')}, \quad i = 1, ..., d-1. \tag{18}$$

The case $d = 2$.

It has been proved by Olsen et al. (1976, p. 880) that in this case z_1 and z_2 constitute a set of sufficient and complete statistics, and in consequence there exists the uniformly minimum variance invariant unbiased estimator (UMVIUE) for σ_1^2 of the form

$$\bar{\sigma}_1^2 = \frac{1}{\alpha_1}(z_1 - z_2). \tag{19}$$

Remark 4.2. In Baksalary et al. [1990, Corollary 1(i) and Corollary 2(i)] a full characterization of designs for which $d = 2$ is given. It is shown that in this case $\operatorname{rank}(\mathbf{N}) = b - \operatorname{rank}(\mathbf{C_\Delta})$ (in that paper the symbol $\mathbf{S}_r$ instead of $\mathbf{C_\Delta}$ is used) and, following Remark 4.1, $\mathcal{R}(\mathbf{D}') \cap \mathcal{N}(\mathbf{\Delta}) = \mathcal{R}(\mathbf{D}' : \mathbf{\Delta}') \cap \mathcal{N}(\mathbf{\Delta})$ and $\mathbf{\Pi}_2 = \mathbf{M_\Delta D'C_\Delta^+ DM_\Delta}$.

COROLLARY 4.1. *If the number d of different eigenvalues of $\mathbf{C}_\Delta$ is 2, then $\hat{\sigma}_1^2$ is admissible for σ_1^2 in the overall model, with (2), and coincides with UMVIUE given by (19).*

The case $d = 3$.

LEMMA 4.2. *If the number d of different eigenvalues of $\mathbf{C}_\Delta$ is 3 and rank$(\mathbf{N})$ = $b-$rank$(\mathbf{C}_\Delta)$, then $\hat{\sigma}_1^2$ is admissible for σ_1^2 in the overall model, with (2), and coincides with $\bar{\gamma}(u,v)$ given by (11) where u and v are such that*

$$u^2 + v = \frac{\nu_3 + \text{rank}(\mathbf{\Pi}_2)}{\nu_3 \alpha_1 \alpha_2}. \tag{20}$$

Proof. If rank$(\mathbf{N}) = b - \text{rank}(\mathbf{C}_\Delta)$, then, from Remark 4.1 and (17), to prove the lemma it is sufficient to show that for a pair (u, v), $u \geq 0$, $v \geq 0$ satisfying (20) we have

$$\lambda_1 \alpha_1 + \lambda_2 = \frac{1 + 2u\alpha_1 + (u^2 + v)\alpha_1^2}{\text{tr}(\mathbf{D\Pi}_2\mathbf{D}')} \tag{21}$$

and

$$\lambda_1 \alpha_2 + \lambda_2 = \frac{1 + 2u\alpha_2 + (u^2 + v)\alpha_2^2}{\text{tr}(\mathbf{D\Pi}_2\mathbf{D}')}, \tag{22}$$

with $\lambda_2 = -\text{rank}(\mathbf{\Pi}_2)/\nu_3 \text{tr}(\mathbf{D\Pi}_2\mathbf{D}')$. Taking differences of the both sides of the above equations we find that

$$\lambda_1 = \frac{2u + (u^2 + v)(\alpha_1 + \alpha_2)}{\text{tr}(\mathbf{D\Pi}_2\mathbf{D}')}.$$

Putting again the above to (21) and (22) we get (20). □

Since under the assumption rank$(\mathbf{N}) = b - \text{rank}(\mathbf{C}_\Delta)$ the estimator $\hat{\sigma}_1^2$ coincides with the Henderson III estimator for σ_1^2 (see Remark 4.1), the first part of the lemma is as has been given by Olsen et al. [1976, Proposition 6.2(b)]. The second part of Lemma 4.2 and the proof are presented here using different characterization of the admissible estimators.

The case $d \geq 4$.

LEMMA 4.3. *If $d > 4$, or if $d = 4$ and rank$(\mathbf{N}) = b - $ rank$(\mathbf{C}_\Delta)$, then the estimator $\hat{\sigma}_1^2$ is inadmissible for σ_1^2 in the overall model, with (2).*

Proof. Suppose that $\hat{\sigma}_1^2$ is admissible. First note that the condition $d \geq 4$ contradicts (18) and that, from (17), for some nonnegative u, v

$$(\lambda_1 \alpha_i + \lambda_2) w_i(u, v) = \frac{c_i}{\text{tr}(\mathbf{D\Pi}_2\mathbf{D}')}, \quad i = 1, 2, ..., d-1, \tag{23}$$

with $\lambda_2 = -\text{rank}(\mathbf{\Pi}_2)/[\nu_d \text{tr}(\mathbf{D\Pi}_2\mathbf{D}')]$. If rank$(\mathbf{N}) = b - \text{rank}(\mathbf{C}_\Delta)$, then $c_i = 1$ for each i. If $c_{i_0} = 0$ for some $i = i_0$, then

$$\lambda_1 = \frac{-\lambda_2}{\alpha_{i_0}}. \tag{24}$$

It follows that we have at most one such i_0, and then from (23)

$$\lambda_1(\alpha_i - \alpha_{i_0})w_i(u,v) = \frac{1}{\text{tr}(\mathbf{D\Pi_2D'})}, \quad i \neq i_0, \quad i \leq d-1.$$

Thus $\alpha_i > \alpha_{i_0}$, i.e. $i_0 = d-1$, and from (23) we find that

$$(u^2+v)\alpha_i^2 + [2u - \lambda_1 \text{tr}(\mathbf{D\Pi_2D'})]\alpha_i + 1 - \text{tr}(\mathbf{D\Pi_2D'})\lambda_2 = 0$$

at least for $i = 1, 2, 3$, which, as having at most two solutions for α_i, is in contradiction with the assumptions. □

Remark 4.3. Since, from (3), $v - \text{rank}(\mathbf{C}_1) = b - \text{rank}(\mathbf{C}_\Delta)$, the condition $\text{rank}(\mathbf{N}) = b - \text{rank}(\mathbf{C}_\Delta)$ is equivalent to the condition $\text{rank}(\mathbf{N}) = v - \text{rank}(\mathbf{C}_1)$, which holds iff $\text{rank}(\mathbf{C}_1)$ is equal to the multiplicity of the unit eigenvalue of $\mathbf{C}_1$ with respect to $\mathbf{\Delta\Delta'}$, i.e., iff the design is orthogonal (cf. Caliński 1993, Corollary 2.1). If, in addition, $d = 2$, i.e., $\mathbf{C}_\Delta$ has only one distinct positive eigenvalue, then it implies and is implied by the fact that the orthogonal design is also proper, i.e. of equal block sizes, whether connected or not (as it follows from Caliński, 1993, Section 3). Thus the considerations can be summarized as follows.

COROLLARY 4.2. *If the design is orthogonal and proper, then the inter-block estimator $\hat{\sigma}_1^2$ is admissible for σ_1^2 in the overall model, with* (2), *and coincides with UMVIUE.*

5. Examples

In this section we consider in details two examples of block designs for which $\text{rank}(\mathbf{N}) = b - \text{rank}(\mathbf{C}_\Delta)$ and $d > 4$. On account of Remark 4.1 and Lemma 4.3, in such a case the estimator (6) of σ_1^2 , being then of the form

$$\hat{\sigma}_1^2 = \frac{1}{\text{tr}(\mathbf{C}_\Delta)}[\sum_{i=1}^{d-1} \nu_i z_i - \text{rank}(\mathbf{C}_\Delta)z_d],$$

is inadmissible, i.e., there exists a uniformly better competitor for $\hat{\sigma}_1^2$.

We shall compare $\hat{\sigma}_1^2$ with two admissible estimators $\bar{\sigma}_1^2(u)$ and $\bar{\sigma}_1^2(\infty)$, from which the first is a Bayes IQUE with respect to prior distribution τ at a given u and $v = 0$ [locally best at $\sigma_1^2 = u$ and $\sigma^2 = 1$, where u is chosen such that $\bar{\sigma}_1^2(u)$ dominates

$\hat{\sigma}_1^2]$ while the second is the limiting Bayes IQUE given by (12) and (15) as

$$\bar{\sigma}_1^2(\infty) = \frac{1}{\text{rank}(\mathbf{C}_\Delta)}[\sum_{i=1}^{d-1} \frac{\nu_i}{\alpha_i} z_i - \text{tr}(\mathbf{C}_\Delta^+) z_d].$$

We would like to pay particular attention to $\bar{\sigma}_1^2(\infty)$ as a practically useful estimator of σ_1^2 because of its "flat" variance function.

5.1. Example 1

Consider a design with the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 2 & 4 & 2 & 4 & 6 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 & 3 \end{bmatrix}. \qquad (25)$$

It can be checked that

i	1	2	3	4	5
α_i	10.4537	8	5.1019	4	0
ν_i	1	1	1	1	29

$d = 5$, $\text{rank}(\mathbf{C}_\Delta) = 4$, $\text{tr}(\mathbf{C}_\Delta) = 27.5556$, $tr(\mathbf{C}_\Delta^+) = 0.6667$ and

$$\hat{\sigma}_1^2 = \frac{1}{27.5556}(z_1 + z_2 + z_3 + z_4 - 4z_5),$$

$$\bar{\sigma}_1^2(\infty) = \frac{1}{4}(\frac{z_1}{10.4537} + \frac{z_2}{8} + \frac{z_3}{5.1019} + \frac{z_4}{4} - 0.6667 z_5).$$

Note,however, that the design represented by (25) is orthogonal and connected. As such, it provides the BLUE under the overall model, with (2), for any contrast $\mathbf{c}'\boldsymbol{\tau}$. The variance of the BLUE, $\widehat{\mathbf{c}'\boldsymbol{\tau}}$, is then of the form $\text{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{c}'(\boldsymbol{\Delta}\boldsymbol{\Delta}')^{-1}\mathbf{c}\sigma^2$, not involving σ_1^2 [as it follows from Corollary 3.3(a) of Caliński and Kageyama (1991)]. Thus, the estimation of σ_1^2 is of no practical use here. It is included for comparative reasons only.

5.2. Example 2

Next, consider a design with the incidence matrix $\mathbf{N} = \text{diag}\{\mathbf{N}(1), \mathbf{N}(2), \mathbf{N}(3)\}$, where $\mathbf{N}(1)$ is as given in (25), while

$$\mathbf{N}(2) = \begin{bmatrix} 9 & 9 \\ 9 & 9 \end{bmatrix}, \qquad \mathbf{N}(3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It can be easily checked that

i	1	2	3	4	5	6	7
α_i	18	10.4537	8	5.1019	4	3	0
ν_i	1	1	1	1	1	2	66

$d = 7$, $\text{rank}(\mathbf{C}_\Delta) = 7$, $\text{tr}(\mathbf{C}_\Delta) = 51.5556$, $\text{tr}(\mathbf{C}_\Delta^+) = 1.3889$ and

$$\hat{\sigma}_1^2 = \frac{1}{51.5556}(z_1 + z_2 + z_3 + z_4 + z_5 + 2z_6 - 7z_7),$$

$$\bar{\sigma}_1^2(\infty) = \frac{1}{7}\left(\frac{z_1}{18} + \frac{z_2}{10.4537} + \frac{z_3}{8} + \frac{z_4}{5.1019} + \frac{z_5}{4} + \frac{2z_6}{3} - 1.3889z_7\right).$$

Here note that the design represented by $\mathbf{N} = \text{diag}\{\mathbf{N}(1), \mathbf{N}(2), \mathbf{N}(3)\}$ is orthogonal and disconnected, composed of the subdesigns, each of them being orthogonal and connected, the first non-proper. This implies that under the overall model, with (2), the BLUEs exist for all contrasts within the subdesigns and also for the contrast between treatments of the second and those of the third subdesign [on account of Corollary 3.3(a), (b) and Remark 3.7 of Caliński and Kageyama (1991)]. While the variances of the former BLUEs involve only σ^2, for which $\hat{\sigma}^2$ is admissible, the variance of the latter contrast involves both σ^2 and σ_1^2 [cf. formula (3.29) in that paper], $\hat{\sigma}_1^2$ being not admissible for σ_1^2. In fact, by Corollary 4.2, $\hat{\sigma}_1^2$ would become admissible if $\mathbf{N}(1)$ were proper.

For the two examples, Tables 1 and 3 show the variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ and $\bar{\sigma}_1^2(\infty)$, and the attainable lower bound (ALB) of the variances, as functions of σ_1^2 at a given $\sigma^2 = 1$. The values of ALB are defined as variances of the locally best invariant unbiased estimators, calculated separately at each σ_1^2 and at $\sigma^2 = 1$, i.e., at $u = \sigma_1^2$ and $v = 0$, where σ_1^2 runs from 0 to 100. Tables 2 and 4 show the loss $l(\sigma_1^2)$ of the variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ and $\bar{\sigma}_1^2(\infty)$ in comparison with the attainable lower bound. This loss is defined according to the formula

$$l(\sigma_1^2) = \frac{\text{var}(\sigma_1^2) - \text{ALB}(\sigma_1^2)}{\text{ALB}(\sigma_1^2)} 100\%.$$

As we can see, for both examples the estimator $\bar{\sigma}_1^2(u)$, with appropriately choosen u, uniformly dominates $\hat{\sigma}_1^2$, while $\bar{\sigma}_1^2(\infty)$ is dominated by $\hat{\sigma}_1^2$ for small σ_1^2, but becomes better for large values of σ_1^2.

Table 1. The variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ for $u = 0.162$, $\bar{\sigma}_1^2(\infty)$ and ALB, as functions of σ_1^2 (Example 1).

σ_1^2	$var(\hat{\sigma}_1^2)$	$var(\bar{\sigma}_1^2(u))$	$var(\bar{\sigma}_1^2(\infty))$	ALB
0.000	0.011989	0.011987	0.017627	0.010400
0.005	0.012729	0.012727	0.018473	0.011231
0.050	0.020665	0.020663	0.027211	0.019881
0.250	0.083724	0.083653	0.090544	0.083153
0.500	0.226349	0.226032	0.225961	0.218585
0.750	0.439863	0.439123	0.423877	0.416517
1.000	0.724267	0.722927	0.684294	0.676945
1.500	1.505745	1.502672	1.392627	1.385293
2.000	2.570780	2.565266	2.350960	2.343635
3.000	5.551528	5.539005	5.017626	5.010311
4.000	9.666510	9.644144	8.684293	8.676983
5.000	14.915727	14.880683	13.350959	13.343653
10.000	58.175330	58.034372	51.684291	51.676991
50.000	1425.063287	1421.523666	1258.350944	1258.343651
100.000	5685.701076	5671.534726	5016.684260	5016.676968

Table 2. The loss $l(\sigma_1^2)$ of the variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ for u=0.162 and $\bar{\sigma}_1^2(\infty)$ in comparison with ALB (Example 1).

σ_1^2	$l(\hat{\sigma}_1^2)$	$l(\bar{\sigma}_1^2(u))$	$l(\bar{\sigma}_1^2(\infty))$
0.000	15.275	15.256	69.488
0.005	13.340	13.326	64.487
0.050	3.942	3.934	36.867
0.250	0.687	0.602	8.888
0.500	3.552	3.407	3.374
0.750	5.605	5.427	1.767
1.000	6.991	6.792	1.086
1.500	8.695	8.473	0.529
2.000	9.692	9.457	0.313
3.000	10.802	10.552	0.146
4.000	11.404	11.146	0.084
5.000	11.781	11.519	0.055
10.000	12.575	12.302	0.014
50.000	13.249	12.968	0.001
100.000	13.336	13.054	0.000

Table 3. The variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ for $u = 0.139$, $\bar{\sigma}_1^2(\infty)$ and ALB, as functions of σ_1^2 (Example 2).

σ_1^2	$var(\hat{\sigma}_1^2)$	$var(\bar{\sigma}_1^2(u))$	$var(\bar{\sigma}_1^2(\infty))$	ALB
0.000	0.005826	0.005810	0.015520	0.003840
0.005	0.006224	0.006211	0.016094	0.004364
0.050	0.010754	0.010735	0.021903	0.009878
0.250	0.051432	0.050785	0.061721	0.049359
0.500	0.149456	0.146634	0.143638	0.131228
0.750	0.299899	0.293356	0.261268	0.248860
1.000	0.502760	0.490952	0.414613	0.402213
1.500	1.065738	1.038765	0.828445	0.816062
2.000	1.838390	1.790072	1.385134	1.372763
3.000	4.012716	3.903170	2.927084	2.914727
4.000	7.025738	6.830247	5.040463	5.028115
5.000	10.877456	10.571301	7.725270	7.712928
10.000	42.716484	41.486248	29.720734	29.708405
50.000	1052.255042	1021.386231	719.970163	719.957845
100.000	4201.244075	4077.712226	2868.496234	2868.483918

Table 4. The loss $l(\sigma_1^2)$ of variances of $\hat{\sigma}_1^2$, $\bar{\sigma}_1^2(u)$ for u=0.139 and $\bar{\sigma}_1^2(\infty)$ in comparison with ALB (Example 2).

σ_1^2	$l(\hat{\sigma}_1^2)$	$l(\bar{\sigma}_1^2(u))$	$l(\bar{\sigma}_1^2(\infty))$
0.000	51.732	51.322	304.207
0.005	42.642	42.339	268.822
0.050	8.863	8.678	121.734
0.250	4.200	2.890	25.047
0.500	13.890	11.739	9.456
0.750	20.509	17.880	4.986
1.000	24.998	22.063	3.083
1.500	30.595	27.290	1.517
2.000	33.919	30.399	0.901
3.000	37.670	33.912	0.424
4.000	39.729	35.841	0.246
5.000	41.029	37.060	0.160
10.000	43.786	39.645	0.042
50.000	46.155	41.868	0.002
100.000	46.462	42.156	0.000

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Received 6 June 1998

O dopuszczalności estymatorów wewnątrzblokowego i międzyblokowego komponentów wariancyjnych

Streszczenie

W pracy rozważane jest zagadnienie dopuszczalności dwóch rodzajów estymatorów komponentów wariancyjnych występujących w modelu randomizacyjnym stosowanym do doświadczeń blokowych. Udowodniono, że estymator wewnątrzblokowy wariancji błędów jest dopuszczalny w klasie estymatorów nieobciążonych. Podano warunki dostateczne, przy których estymator międzyblokowy wariancji efektów blokowych jest dopuszczalny. Warunki te wyrażone są poprzez funkcje macierzy incydencji układu. Przedstawiono przykłady modeli, w których estymator międzyblokowy nie jest dopuszczalny. Dla takich wypadków pokazano, jak można poprawić jednostajnie ten estymator poprzez użycie jednego z estymatorów dopuszczalnych.

Słowa kluczowe: dopuszczalność, estymacja wewnątrzblokowa i międzyblokowa, estymatory komponentów wariancyjnych, niezmiennicze kwadratowe estymatory dopuszczalne, układy blokowe.